

7.1. DEFINITION OF THE LAPLACE TRANSFORM.

We can view differentiation and integration as two **transforms**. They transform a function into another function.

For example

$$\frac{d}{dx} x^n = n x^{n-1}, \quad \frac{d}{dx} e^x = e^x$$

$$\int x^4 dx = \frac{1}{5} x^5 + c, \quad \int \cos x dx = \sin x + c.$$

Moreover, these two transforms are **linear**:

$$\frac{d}{dx} [\alpha f(x) + \beta g(x)] = \alpha \frac{d}{dx} f(x) + \beta \frac{d}{dx} g(x)$$

$$\int [\alpha f(x) + \beta g(x)] dx = \alpha \int f(x) dx + \beta \int g(x) dx.$$

In this section, we will examine a special type of integral transform called the **Laplace transform**.

In addition to possessing the linearity property the Laplace transform has many other interesting properties that make it very useful in solving linear IVP's.

INTEGRAL TRANSFORM

If $f(x, y)$ is a function of two variables, then a definite integral of f with respect to one of the variables leads to a function of the other variable.

For example, $\int_1^2 2xy^2 dx = 3y^2$.

Similarly, a definite integral such that $\int_a^b k(s, t) f(t) dt$ transforms a function f of the variable t into a function F of the variable s . In particular, we are interested in the following **integral transform**:

$$\int_0^{\infty} K(s, t) f(t) dt = \lim_{b \rightarrow \infty} \int_0^b K(s, t) f(t) dt. \quad (1)$$

If the limit in (1) exists, we say that the integral is **convergent**, and is **divergent** otherwise.

DEFINITION

The function $K(s, t)$ in (1) is called the **kernel** of the transform. The choice $K(s, t) = e^{-st}$ as a kernel gives us an especially important transform:

Defn. (Laplace Transform)

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (2)$$

is said to be the **Laplace transform** of f , provided that the integral converges.

Example 1. Evaluate $\mathcal{L}\{1\}$.

Solution:
$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^{\infty} e^{-st} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \left. \frac{-e^{-st}}{s} \right|_0^b = \lim_{b \rightarrow \infty} \frac{1}{s} (-e^{-sb} + 1) \\ &= \frac{1}{s}\end{aligned}$$

provided that $s > 0$. The integral diverges when $s \leq 0$.

□

Example 2. Evaluate $\mathcal{L}\{t\}$.

Solution: $\mathcal{L}\{t\} = \int_0^{\infty} t e^{-st} dt$. Integrating by parts and using $\lim_{t \rightarrow \infty} t e^{-st} = 0$, for $s > 0$, we get

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^{\infty} \underbrace{t}_{u} \underbrace{e^{-st}}_{dv} dt \\ &= \underbrace{\frac{-te^{-st}}{s}}_{uv} \Big|_0^{\infty} - \int_0^{\infty} \underbrace{\frac{-e^{-st}}{s}}_v \underbrace{dt}_{du} \\ &= \lim_{t \rightarrow \infty} (-te^{-st} - 0) + \frac{1}{s} \int_0^{\infty} e^{-st} dt \\ &= 0 + \frac{1}{s} \left(\frac{1}{s} \right) \quad (\text{by Example 1}) \\ &= \frac{1}{s^2}\end{aligned}$$

□

Example 3: Evaluate $\mathcal{L}\{e^{at}\}$ ($a \neq 0$).

Solution:

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^{\infty} e^{at} e^{-st} dt \\&= \int_0^{\infty} e^{-(s-a)t} dt \\&= \left. \frac{-e^{-(s-a)t}}{s-a} \right|_0^{\infty} \\&= \frac{1}{s-a}\end{aligned}$$

given that $s > a$. For $s \leq a$, the integral diverges. $\square$

Example 4: Evaluate $\mathcal{L}\{\sin 2t\}$.

Solution $\mathcal{L}\{\sin 2t\} = \int_0^{\infty} e^{-st} \sin 2t dt$

$$= \int_0^{\infty} \underbrace{\sin 2t}_u \underbrace{e^{-st}}_{dv} dt$$

$(\lim_{t \rightarrow \infty} e^{-st} \sin 2t = 0 \text{ for } s > 0)$

$$= \left. \frac{-e^{-st} \sin 2t}{s} \right|_0^{\infty} + \frac{2}{s} \int_0^{\infty} e^{-st} \cos 2t dt$$

$$= \frac{2}{s} \int_0^{\infty} e^{-st} \cos 2t dt, s > 0$$

$$= \frac{2}{s} \left[\left. \frac{-e^{-st} \cos 2t}{s} \right|_0^{\infty} - \frac{2}{s} \int_0^{\infty} e^{-st} \sin 2t dt \right]$$

$(\lim_{t \rightarrow \infty} e^{-st} \cos 2t = 0 \text{ for } s > 0)$

$$= \frac{2}{s^2} - \frac{4}{s^2} \mathcal{L}\{\sin 2t\}$$

Thus $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2 + 4}, s > 0$ $\square$

$\mathcal{L}$ IS A LINEAR TRANSFORM.

For a linear combination of functions f, g , we can write:

$$\int_0^{\infty} e^{-st} [\alpha f(t) + \beta g(t)] dt = \alpha \int_0^{\infty} e^{-st} f(t) dt + \beta \int_0^{\infty} e^{-st} g(t) dt.$$

Whenever both integrals converge for $s > c$, for some constant c . Hence it follows that

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} \quad (3). \\ = \alpha F(s) + \beta G(s)$$

Example:

$$\begin{aligned} \text{a) } \mathcal{L}\{1+5t\} &= \mathcal{L}\{1\} + 5\mathcal{L}\{t\} \\ &= \frac{1}{s} + \frac{5}{s^2} = \frac{s+5}{s^2} \quad (s>0) \end{aligned}$$

$$\begin{aligned} \text{b) } \mathcal{L}\{4e^{5t} - 10\sin 2t\} \\ &= 4\mathcal{L}\{e^{5t}\} - 10\mathcal{L}\{\sin 2t\} \\ &= \frac{4}{s-5} - \frac{20}{s^2+4} \end{aligned}$$

$$\begin{aligned} \text{c) } \mathcal{L}\{20e^{-3t} + 7t - 9\} \\ &= 20\mathcal{L}\{e^{-3t}\} + 7\mathcal{L}\{t\} - 9\mathcal{L}\{1\} \\ &= \frac{20}{s+3} + \frac{7}{s^2} - \frac{9}{s} \end{aligned}$$

Theorem: (Laplace transforms of Some Basic Functions)

$$a) \mathcal{L}\{1\} = \frac{1}{s} \quad (s > 0)$$

$$b) \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad n=1, 2, 3, \dots \quad (s > 0)$$

$$c) \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (s > a)$$

$$d) \mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2} \quad (s > 0)$$

$$e) \mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2} \quad (s > 0)$$

$$f) \mathcal{L}\{\sinh kt\} = \frac{k}{s^2 - k^2} \quad (s > 0)$$

$$g) \mathcal{L}\{\cosh kt\} = \frac{s}{s^2 - k^2} \quad (s > k)$$

SUFFICIENT CONDITIONS FOR EXISTENCE OF $\mathcal{L}$

We say f is **piecewise continuous** on $[0, \infty)$ if, in any interval $0 \leq a \leq t \leq b$, there are at most a finite number of points t_k , $k=1, 2, \dots, n$ at which f is discontinuous.

A function f is said to be **exponential order** if there exist constants c , $M > 0$, and $T > 0$ s.t.
 $|f(t)| \leq M e^{ct}$ for all $t > T$.

Theorem: If f is piecewise continuous on $[0, \infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exist for $s > c$.

Proof.

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^T e^{-st} f(t) dt + \int_T^\infty e^{-st} f(t) dt \\ &= I_1 + I_2\end{aligned}$$

I_1 exists because it can be written as a sum of integrals over intervals on which $e^{-st} f(t)$ is continuous.

Now since f is of exponential order, $\exists c, M > 0, T > 0$ s.t

$$|f(t)| \leq M e^{ct} \text{ for } t > T.$$

$$\begin{aligned}\Rightarrow |I_2| &\leq \int_T^\infty |e^{-st} f(t)| dt \leq M \int_T^\infty e^{-(s-c)t} dt \\ &= M \frac{e^{-(s-c)T}}{s-c} \text{ for } s > c.\end{aligned}$$

$\Rightarrow I_2$ exists for $s > c$.

$\Rightarrow \mathcal{L}\{f(t)\}$ exists for $s > c$. $\square$

Example 1: Let $f(t) = \begin{cases} 0 & 0 \leq t < 3 \\ 2 & t \geq 3. \end{cases}$

Find $\mathcal{L}\{f(t)\}$.

Solution:

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^3 0 \cdot e^{-st} dt + \int_3^\infty e^{-st} \cdot 2 dt \\ &= 2 \frac{e^{-st}}{-s} \Big|_3^\infty = \frac{2e^{-3s}}{s}, s > 0 \quad \square\end{aligned}$$

Theorem: If f is piecewise continuous on $[0, \infty)$ and of exponential order and $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s \rightarrow \infty} F(s) = 0$.

Proof: $\exists \delta, M_1 > 0, T > 0$ s.t.
 $|f(t)| \leq M_1 e^{\delta t}$ for $t > T$.

Since $f(t)$ is piecewise continuous $\Rightarrow f(t)$ is bounded on the interval $[0, T]$.

$\Rightarrow \exists M_2$ s.t. $|f(t)| \leq M_2 = M_2 e^{0t}$. Take $M = \max\{M_1, M_2\}$, $c = \max\{0, \delta\}$, then

$$\begin{aligned} |F(s)| &\leq \int_0^{\infty} e^{-st} |f(t)| dt \leq M \int_0^{\infty} e^{-(s-c)t} dt \\ &= \frac{M}{s-c} \rightarrow 0 \text{ as } s \rightarrow \infty. \end{aligned}$$

□

